昭和薬科大学紀要
Vol.52；1-9，2018

原 著

Ill-conditioned simultaneous linear equations to which the regularized solution almost coincides with the exact one

Takemi SHIGETA

Professor, Laboratory of Applied Mathematics, Showa Pharmaceutical University

Abstract

Ill-conditioned simultaneous linear equations in two unknowns are considered. A small noise contained in the non-homogeneous term of the ill-conditioned equations retains the potential to affect sensitively on the accuracy of the solution. The truncated singular value decomposition can then be applied to the equations to obtain a regularized solution of which the change is sufficiently small for a small change of the non-homogeneous term. In this study, the non-homogeneous term of the equations such that the regularized solution almost coincides with the exact one is defined to be *exact*. The purpose of the study is to reconstruct the unknown *exact* non-homogeneous term from the regularized solution. Simple numerical experiments show that the exact solution to the equations with the reconstructed non-homogeneous term almost coincides with the regularized one to the same equations. It is concluded that the *exact* non-homogeneous term can successfully be reconstructed.

Keywords
truncated singular value decomposition, regularized solution, Moore-Penrose pseudo-inverse matrix, exact non-homogeneous term

1 Introduction

We consider the following simultaneous linear equations[4)]:

$$\begin{cases} 0.4x_1 + 1.2x_2 = 5.2 \\ 3.5x_1 + 10.501x_2 = 45.504 \end{cases} \tag{1}$$

and

$$\begin{cases} 0.4x_1 + 1.2x_2 = 5.2 \\ 3.5x_1 + 10.501x_2 = 45.501 \end{cases} . \tag{2}$$

The solution to Eq. (1) is $x_1 = 1, x_2 = 4$, while the solution to Eq. (2) is $x_1 = 10, x_2 = 1$. We can see that a small change of the right-hand side called the non-homogeneous term encompasses a large change of the solution.

The Cauchy problem of the Laplace equation[5)], known as an inverse problem of partial differential equations, is an ill-posed problem[2)]. The resultant simultaneous linear equations after discretization of the Cauchy problem have the same issue as Eq. (1) (or Eq. (2)). Namely, a small noise contained in the non-homogeneous term of the resultant equations affects sensitively on the accuracy of the solution. The non-homogeneous term consists of the observed data called the Cauchy data. For the Cauchy data without noises, an accurate numerical solution to the Cauchy problem can be obtained from the solution to the resultant simultaneous equations. On the other hand, for the noisy Cauchy data, a numerical solution to the Cauchy problem drastically jumbles due to a small noise contained in the non-homogeneous term. Then, applying a regularization method[1)] to the resultant simultaneous linear equations, we can obtain an accurate numerical solution to the Cauchy problem[3)].

Consequently, when we consider simultaneous linear equations like Eq. (1) (or Eq. (2)), we notice that the non-homogeneous term can be regarded as an *exact* one which does not contain noises if the solution obtained without applying the regularization method almost coincides with the solution obtained by applying the regularization method.

We heretofore have no criterion for the correctness of the non-homogeneous term of arbitrarily given simultaneous linear equations. In this study, we will define an *exact* non-homogeneous term and propose how to reconstruct the *exact* one by using the truncated singular value decomposition as a regularization method.

2 Problem setting

For simplicity, we consider the following simultaneous linear equations:

$$x_1 + ax_2 = b, \tag{3}$$

$$x_1 + (a+\varepsilon)x_2 = b + \delta, \tag{4}$$

where a, b are given real constants, $\varepsilon > 0$ a given sufficiently small constant, and $\delta > 0$ a given perturbed parameter. The simultaneous linear equations (3), (4) are written in the matrix form:

$$A\boldsymbol{x} = \boldsymbol{b}(\delta),$$

where the coefficient matrix A and the both of the solution $\boldsymbol{x}$ and the non-homogeneous term $\boldsymbol{b}$ depending on δ are denoted by

$$A := \begin{pmatrix} 1 & a \\ 1 & a+\varepsilon \end{pmatrix}, \quad \boldsymbol{x} = \boldsymbol{x}(\delta) := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{b} = \boldsymbol{b}(\delta) := \begin{pmatrix} b \\ b+\delta \end{pmatrix}.$$

Since A is not singular ($|A| = \varepsilon \neq 0$), the solution to Eqs. (3), (4) is obtained as

$$\boldsymbol{x} = \boldsymbol{x}(\delta) = \begin{pmatrix} b \\ 0 \end{pmatrix} + \frac{\delta}{\varepsilon}\begin{pmatrix} -a \\ 1 \end{pmatrix}. \tag{5}$$

If δ changes to δ', then the changes of the non-homogeneous term and the solution are given by

$$\Delta\boldsymbol{b} := \boldsymbol{b}(\delta') - \boldsymbol{b}(\delta) = \begin{pmatrix} 0 \\ \delta' - \delta \end{pmatrix},$$

$$\Delta\boldsymbol{x} := \boldsymbol{x}(\delta') - \boldsymbol{x}(\delta) = \frac{\delta' - \delta}{\varepsilon}\begin{pmatrix} -a \\ 1 \end{pmatrix},$$

respectively. A small change of the non-homogeneous term ($\|\Delta\boldsymbol{b}\| = |\delta' - \delta|$) encompasses a large change of the solution if $\delta' - \delta \geq \varepsilon$. Such simultaneous linear equations are called ill-conditioned. Therefore, we need to obtain the solution which does not drastically change for a small change of the non-homogeneous term.

3 Singular value decomposition (SVD)

Let A^T denote the transpose of the matrix A. We consider the following eigenvalue problem of the matrix A^TA:

$$A^TA\boldsymbol{v} = \lambda\boldsymbol{v}, \quad \boldsymbol{v} \neq \boldsymbol{0}, \tag{6}$$

where λ denotes the eigenvalue of A^TA and $\boldsymbol{v}$ the eigenvector associated with λ. Eq. (6) is rewritten as

$$(A^TA - \lambda I)\boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{v} \neq \boldsymbol{0}. \tag{7}$$

Since Eq. (7) has a non-zero solution $\boldsymbol{v}$, we know that the determinant of $A^TA - \lambda I$ is zero. Hence, we obtain the characteristic equation of A:

$$|A^TA - \lambda I| = \lambda^2 - (a^2 + (a+\varepsilon)^2 + 2)\lambda + \varepsilon^2 = 0. \tag{8}$$

The eigenvalues λ_1, λ_2 ($\lambda_1 > \lambda_2$) are the solutions to Eq. (8), which satisfies

$$\lambda_1 + \lambda_2 = a^2 + (a+\varepsilon)^2 + 2 = 2(a^2+1) + O(\varepsilon), \tag{9}$$

$$\lambda_1\lambda_2 = \varepsilon^2, \tag{10}$$

where we assume $\varepsilon \to 0$ to derive the asymptotic expansion although ε is a sufficiently small constant. The eigenvalue λ_1 is expressed in the form:

$$\begin{aligned}\lambda_1 &= \frac{\{a^2 + (a+\varepsilon)^2 + 2\} + \sqrt{\{a^2 + (a+\varepsilon)^2 + 2\}^2 - 4\varepsilon^2}}{2} \\ &= 2(a^2 + 1 + a\varepsilon) + O(\varepsilon^2) = 2(a^2+1) + O(\varepsilon)\end{aligned}$$

(see Section A). Eq. (10) implies

$$\lambda_2 = \frac{\varepsilon^2}{\lambda_1} = \frac{\varepsilon^2}{2(a^2+1) + O(\varepsilon)} = \frac{\varepsilon^2}{2(a^2+1)} + O(\varepsilon^3).$$

The normalized eigenvectors associated with the eigenvalues λ_1, λ_2 are given as

$$\begin{aligned}\boldsymbol{v}_1 &:= \frac{1}{\sqrt{a^2+1+O(\varepsilon)}}\begin{pmatrix}1+O(\varepsilon)\\ a+O(\varepsilon)\end{pmatrix} = \frac{1}{\sqrt{a^2+1}}\begin{pmatrix}1\\ a\end{pmatrix} + O(\varepsilon), \\ \boldsymbol{v}_2 &:= \frac{1}{\sqrt{a^2+1+O(\varepsilon)}}\begin{pmatrix}a+O(\varepsilon)\\ -1+O(\varepsilon)\end{pmatrix} = \frac{1}{\sqrt{a^2+1}}\begin{pmatrix}a\\ -1\end{pmatrix} + O(\varepsilon),\end{aligned}$$

respectively, where we abbreviate a vector or a matrix which consists of components $O(\varepsilon)$ as $O(\varepsilon)$ for simplicity. Since we see from $(A^TA)^T = A^T(A^T)^T = A^TA$ that A^TA is symmetric, the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of A^TA are orthonormal.

Dividing Eq. (6) by $\sigma_i := \sqrt{\lambda_i}$, we have

$$A^T\left(\frac{1}{\sigma_i}A\boldsymbol{v}_i\right) = \sigma_i\boldsymbol{v}_i.$$

Let

$$\boldsymbol{u}_i := \frac{1}{\sigma_i}A\boldsymbol{v}_i,$$

which can be calculated as

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\varepsilon),$$
$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O(\varepsilon).$$

Then, we have

$$A^T \boldsymbol{u}_i = \sigma_i \boldsymbol{v}_i, \qquad A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i. \tag{11}$$

Multiplying the first equation of Eq. (11) by the matrix A from the left side, we obtain

$$AA^T \boldsymbol{u}_i = A(\sigma_i \boldsymbol{v}_i) = \sigma_i A\boldsymbol{v}_i = \sigma_i(\sigma_i \boldsymbol{u}_i) = \lambda_i \boldsymbol{u}_i,$$

from which we know that $\boldsymbol{u}_i$ is the eigenvector associated with the eigenvalue λ_i of AA^T. Since AA^T is symmetric, $\boldsymbol{u}_1, \boldsymbol{u}_2$ are orthonormal. We can confirm that $\boldsymbol{u}_1, \boldsymbol{u}_2$ are orthonormal also in this way:

$$\begin{aligned}
(\boldsymbol{u}_i, \boldsymbol{u}_j) &= \left(\frac{1}{\sigma_i} A\boldsymbol{v}_i, \frac{1}{\sigma_j} A\boldsymbol{v}_j \right) = \frac{1}{\sigma_i \sigma_j} (A^T A\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{1}{\sigma_i \sigma_j} (\lambda_i \boldsymbol{v}_i, \boldsymbol{v}_j) \qquad (\because \quad A^T A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i) \\
&= \frac{\sigma_i}{\sigma_j} (\boldsymbol{v}_i, \boldsymbol{v}_j) \qquad (\because \quad \lambda_i = \sigma_i^2) \\
&= \delta_{ij}
\end{aligned}$$

with the Kronecker delta δ_{ij}.

From the second equation of Eq. (11), we have

$$(\sigma_1 \boldsymbol{u}_1 \ \sigma_2 \boldsymbol{u}_2) = (A\boldsymbol{v}_1 \ A\boldsymbol{v}_2)$$

or

$$(\boldsymbol{u}_1 \ \boldsymbol{u}_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = A(\boldsymbol{v}_1 \ \boldsymbol{v}_2).$$

Since $(\boldsymbol{v}_1 \ \boldsymbol{v}_2)$ is an orthogonal matrix, multiplying the above equation by $(\boldsymbol{v}_1 \ \boldsymbol{v}_2)^T$ from the right side, we can obtain the singular value decomposition (SVD) of the matrix A as follows:

$$A = U\Sigma V^T = \sum_{j=1}^{2} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T, \tag{12}$$

where the orthogonal matrices U, V and the diagonal matrix Σ are given by

$$U := (\boldsymbol{u}_1 \ \boldsymbol{u}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + O(\varepsilon),$$
$$V := (\boldsymbol{v}_1 \ \boldsymbol{v}_2) = \frac{1}{\sqrt{a^2+1}} \begin{pmatrix} 1 & a \\ a & -1 \end{pmatrix} + O(\varepsilon),$$
$$\Sigma := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2(a^2+1)} & 0 \\ 0 & \dfrac{\varepsilon}{\sqrt{2(a^2+1)}} \end{pmatrix} + O(\varepsilon),$$

and σ_j denotes the singular value of A, and $\boldsymbol{u}_j, \boldsymbol{v}_j$ the left singular vector and the right singular vector associated with σ_j, respectively.

The condition number corresponding to 2-norm can be expressed as

$$\mathrm{cond}(A) = \frac{\sigma_1}{\sigma_2} = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} = \left(\frac{\lambda_1}{\varepsilon^2/\lambda_1}\right)^{1/2} = \frac{\lambda_1}{\varepsilon} = \frac{2(a^2+1+a\varepsilon)+O(\varepsilon^2)}{\varepsilon}$$
$$= \frac{2(a^2+1)}{\varepsilon} + 2a + O(\varepsilon)$$

with the maximal and the minimal singular values σ_1 and σ_2. We can see that $\mathrm{cond}(A)$ becomes large as the determinant $|A| = \varepsilon > 0$ tends to zero.

4 Regularized solution by the truncated SVD

Based on the SVD, the inverse matrix of A can be expressed in the form:

$$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T = \sum_{j=1}^{2}\frac{1}{\sigma_j}\boldsymbol{v}_j\boldsymbol{u}_j^T \tag{13}$$

by using $VV^T = I, UU^T = I$.

The solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ can be written in the form:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \sum_{j=1}^{2}\frac{(\boldsymbol{u}_j, \boldsymbol{b})}{\sigma_j}\boldsymbol{v}_j$$

and the solution $\boldsymbol{x} + \Delta\boldsymbol{x}$ to $A(\boldsymbol{x}+\Delta\boldsymbol{x}) = \boldsymbol{b} + \Delta\boldsymbol{b}$, that is, $\Delta\boldsymbol{x}$ to $A\Delta\boldsymbol{x} = \Delta\boldsymbol{b}$ can be written in the form:

$$\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b} = \sum_{j=1}^{2}\frac{(\boldsymbol{u}_j, \Delta\boldsymbol{b})}{\sigma_j}\boldsymbol{v}_j,$$

from which we can see that the second term ($j = 2$) of the right-hand side may be large since σ_2 is close to zero. That is the reason why the solution $\boldsymbol{x}$ retains the potential to change drastically for a small change of the non-homogeneous term $\boldsymbol{b}$.

Since the singular value $\sigma_2 = \varepsilon/\sqrt{2(a^2+1)}$ is sufficiently close to zero in Eq. (12), the matrix A can be approximated as

$$A_1 := \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T = \begin{pmatrix} 1 & a \\ 1 & a \end{pmatrix} + O(\varepsilon),$$

which is called the truncated SVD (TSVD). According to Eq. (13), the Moore-Penrose pseudo-inverse matrix of A_1 is defined by

$$A_1^\dagger := \frac{1}{\sigma_1}\boldsymbol{v}_1\boldsymbol{u}_1^T = \frac{1}{2(a^2+1)}\begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix} + O(\varepsilon).$$

Then, a regularized solution can be defined as

$$\boldsymbol{x}_1(\delta) := A_1^\dagger\boldsymbol{b}(\delta) = \frac{(\boldsymbol{u}_1, \boldsymbol{b}(\delta))}{\sigma_1}\boldsymbol{v}_1 = \frac{b}{a^2+1}\begin{pmatrix} 1 \\ a \end{pmatrix} + O(\varepsilon) + O(\delta), \tag{14}$$

which does not have the term with sufficiently small σ_2 and hence does not drastically change for the perturbation of δ.

We now define an *exact* non-homogeneous term by $\boldsymbol{b}(\delta)$ such that the first term of the right-hand side in the regularized solution (14) coincides with the conventional exact one (5). If the first term of the right-hand side in Eq. (14) equals Eq. (5), then we obtain

$$\delta = \delta_1 := \frac{ab\varepsilon}{a^2+1}, \tag{15}$$

which gives the *exact* non-homogeneous term as follows:

$$\boldsymbol{b}(\delta_1) = \frac{b}{a^2+1}\begin{pmatrix} a^2+1 \\ a^2+1+a\varepsilon \end{pmatrix}.$$

5 Least norm solution

The general solution to Eq. (3) is given by

$$\boldsymbol{x}_g = \begin{pmatrix} b \\ 0 \end{pmatrix} + t\begin{pmatrix} -a \\ 1 \end{pmatrix}, \quad \forall t \in \boldsymbol{R}. \tag{16}$$

From

$$\|\boldsymbol{x}_g\|^2 = (b-at)^2 + t^2 = (a^2+1)\left(t - \frac{ab}{a^2+1}\right)^2 + \frac{b^2}{a^2+1},$$

we know that

$$t = \frac{ab}{a^2+1} \tag{17}$$

implies the least norm solution

$$\boldsymbol{x}_0 = \frac{b}{a^2+1}\begin{pmatrix} 1 \\ a \end{pmatrix}, \tag{18}$$

which coincides with the first term of the right-hand side in the regularized solution (14) except the sufficiently small term with respect to ε and δ.

On the other hand, comparing the general solution (16) with the solution (5), we have

$$t = \frac{\delta}{\varepsilon}. \tag{19}$$

From Eqs. (17) and (19), we can easily obtain (15).

In addition, the general solution to Eq. (4) is obtained as

$$\boldsymbol{x}_g = \begin{pmatrix} b \\ 0 \end{pmatrix} + t\begin{pmatrix} -a \\ 1 \end{pmatrix} + \begin{pmatrix} \delta - t\varepsilon \\ 0 \end{pmatrix}, \quad \forall t \in \boldsymbol{R},$$

which coincides with the solution (5) as $\delta - t\varepsilon = 0$. Hence, we can confirm Eq. (19).

6 Solution norm against perturbation

The norm of the solution (5) is written as

$$\|\boldsymbol{x}(\delta)\|^2 = \left(b - \frac{\delta}{\varepsilon}a\right)^2 + \left(\frac{\delta}{\varepsilon}\right)^2 = \frac{a^2+1}{\varepsilon^2}\left(\delta - \frac{ab\varepsilon}{a^2+1}\right)^2 + \frac{b^2}{a^2+1}. \tag{20}$$

Hence, if

$$\delta = \delta_1 = \frac{ab\varepsilon}{a^2+1},$$

then the solution norm $\|\boldsymbol{x}(\delta)\|$ depending on δ is minimized. Consequently, the *exact* non-homogeneous term can be given by $\boldsymbol{b}(\delta_1)$ when we take $\delta = \delta_1$ which minimizes the solution norm $\|\boldsymbol{x}(\delta)\|$.

Putting $\delta = \delta_1$ in Eq. (5), we obtain the exact solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ with the *exact* $\boldsymbol{b} = \boldsymbol{b}(\delta_1)$ as follows:

$$\boldsymbol{x}(\delta_1) = \begin{pmatrix} b \\ 0 \end{pmatrix} + \frac{ab}{a^2+1}\begin{pmatrix} -a \\ 1 \end{pmatrix} = \frac{b}{a^2+1}\begin{pmatrix} 1 \\ a \end{pmatrix}. \tag{21}$$

Then, we can see that the regularized solution (14) can be regarded as an approximation to the exact solution (21) as follows:

$$\boldsymbol{x}_1(\delta_1) = \boldsymbol{x}(\delta_1) + O(\varepsilon) + O(\delta_1).$$

7 Numerical example

Let $a = 5$, $b = 26$ and $\varepsilon = 10^{-3}$. We numerically obtain the conventional exact and the regularized solutions to the simultaneous linear equations (3), (4) for given δ.

Table 1: Regularized solutions vs. conventional exact ones

δ	$\boldsymbol{x}_1(\delta)$	$\boldsymbol{x}(\delta)$
$2\cdot 10^{-3}$	$(0.9998, 4.9997)^T$	$(16, 2)^T$
$5\cdot 10^{-3}$	$(0.9999, 5.0000)^T$	$(1, 5)^T$
$7\cdot 10^{-3}$	$(0.9999, 5.0002)^T$	$(-9, 7)^T$

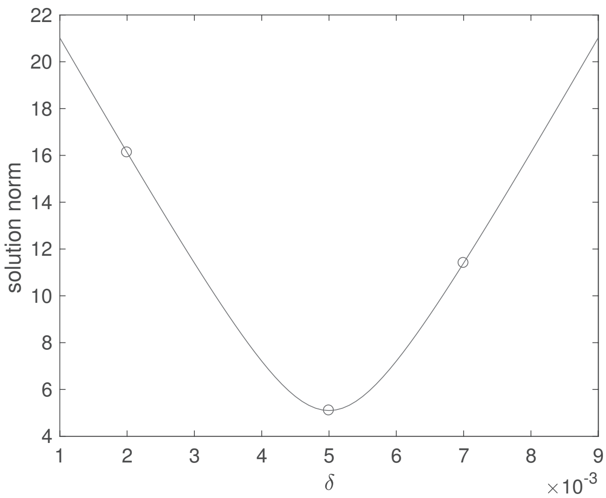

Figure 1: Solution norm against δ

Table 1 shows that the regularized solutions $\boldsymbol{x}_1(\delta)$ for $\delta = 2 \cdot 10^{-3}, 5 \cdot 10^{-3}, 7 \cdot 10^{-3}$ are approximately equal to $(1,5)^T$, while the conventional exact solutions $\boldsymbol{x}(\delta)$ for $\delta = 2 \cdot 10^{-3}, 7 \cdot 10^{-3}$ are $(16,2)^T, (-9,7)^T$, respectively, which are much different from the regularized solutions. The conventional exact solution $\boldsymbol{x}(\delta)$ for $\delta = 5 \cdot 10^{-3}$ is $(1,5)^T$, which almost coincides with the regularized solution. Actually, we can see that if we take $\delta = \delta_1 = 5 \cdot 10^{-3}$ according to Eq. (15), the conventional exact solution almost coincides with the regularized one. We can therefore reconstruct the *exact* non-homogeneous term as $\boldsymbol{b}(5 \cdot 10^{-3}) = (26, 26.005)^T$.

Figure 1 shows the solution norms $\|\boldsymbol{x}(\delta)\|$ for $\delta = 2 \cdot 10^{-3}, 5 \cdot 10^{-3}, 7 \cdot 10^{-3}$ indicated by the small circles and the solid curve obtained by (20) against δ $(1 \cdot 10^{-3} \le \delta \le 9 \cdot 10^{-3})$. We can see from the figure that $\|\boldsymbol{x}(\delta)\|$ is minimized at $\delta = \delta_1 = 5 \cdot 10^{-3}$ which gives the *exact* non-homogeneous term.

8 Conclusions

In this study, we have considered the ill-conditioned simultaneous linear equations in two unknowns. Since the non-homogeneous term of simultaneous linear equations appearing in real engineering problems consists of observed noisy data, the exact non-homogeneous term without noises is unknown. The *exact* non-homogeneous term based on the TSVD is defined and reconstructed in the study. This approach will be extended to reconstruct *exact* data from observed noisy data in ill-conditioned simultaneous linear equations in n unknowns.

A Appendix

For any $p \in \boldsymbol{R}$, the binomial series

$$(x+h)^p = \sum_{k=0}^{\infty} \binom{p}{k} h^k x^{p-k}$$

holds for any h such that $|h| < |x|$ with the generalized binomial coefficients

$$\binom{p}{k} := \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

Then, we have

$$(x+h)^p = x^p + O(h) \qquad (h \to 0). \tag{22}$$

Using Eq. (22), we can derive

$$\begin{aligned}
&\sqrt{\{a^2+(a+\varepsilon)^2+2\}^2-4\varepsilon^2} = \sqrt{\{2(a^2+1+a\varepsilon)+\varepsilon^2\}^2-4\varepsilon^2} \\
&= \sqrt{[\{2(a^2+1+a\varepsilon)\}^2+O(\varepsilon^2)]-4\varepsilon^2} \quad (\because\ x = 2(a^2+1+a\varepsilon),\ h=\varepsilon^2,\ p=2 \text{ in Eq. (22)}) \\
&= \sqrt{\{2(a^2+1+a\varepsilon)\}^2+O(\varepsilon^2)} \\
&= \sqrt{\{2(a^2+1+a\varepsilon)\}^2}+O(\varepsilon^2) \quad (\because\ x = \{2(a^2+1+a\varepsilon)\}^2,\ h=O(\varepsilon^2),\ p=1/2 \text{ in Eq. (22)}) \\
&= 2(a^2+1+a\varepsilon)+O(\varepsilon^2) \quad (\because\ a^2+1+a\varepsilon = (a+\varepsilon/2)^2+1-\varepsilon^2/4 > 0) \qquad (23) \\
&= 2(a^2+1)+O(\varepsilon) \qquad (24)
\end{aligned}$$

as $\varepsilon \to 0$, where the asymptotic expansion (23) is more accurate than the expansion (24). Therefore, we obtain

$$\lambda_1 = \frac{\{a^2 + (a+\varepsilon)^2 + 2\} + \{2(a^2 + 1 + a\varepsilon) + O(\varepsilon^2)\}}{2}$$
$$= 2(a^2 + 1 + a\varepsilon) + O(\varepsilon^2) = 2(a^2 + 1) + O(\varepsilon).$$

References

1) Hansen, P. C.: Analysis of discrete ill-posed problems by means of the L-curve, *SIAM Review*, **34** (4), 561–580 (1992).

2) Kress, R.: Linear Integral Equations, Springer-Verlag (1989).

3) Shigeta, T. and Young, D. L.: Method of fundamental solutions with optimal regularization techniques for the Cauchy problem of the Laplace equation with singular points, *Journal of Computational Physics*, **228** (6), 1903–1915 (2009).

4) Yamamoto, T.: Introduction to Numerical Analysis (*in Japanese*), Saiensu-sha (2003).

5) Isakov, V.: Inverse Problems for Partial Differential Equations, Springer (2006).

正則化解が真の解にほぼ一致する
悪条件連立 1 次方程式

繁田 岳美

昭和薬科大学 応用数学研究室 教授

概 要

悪条件な 2 元連立 1 次方程式を考える．悪条件方程式の非斉次項に混入した微小な誤差が解の精度に敏感に影響を及ぼす可能性がある．その際，打ち切り特異値分解を悪条件方程式に適用することで，非斉次項の微小変化に対して変化が微小であるような正則化解が得られる．本研究では，正則化解が真の解とほぼ一致するような方程式の非斉次項を “正しい” と定義する．本研究の目的は正則化解から未知の “正しい” 非斉次項を復元することである．簡単な数値実験により，復元された非斉次項からなる方程式の真の解は同方程式の正則化解にほぼ一致することが示される．以上より，“正しい” 非斉次項が首尾よく復元できたと結論付けられる．

キーワード
打ち切り特異値分解，正則化解，Moore-Penrose 擬似逆行列，真の非斉次項